

Fitting Spatial Stochastic Frontier Models in Stata

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- Classical production theory assumes that firms formulate optimal production plans in a given market environment. In reality, firms may deviate from their optimal production plans, resulting in output below potential output levels or costs that exceed the minimum cost.
- Economics researchers have proposed the concept of technical efficiency to measure and analyze the extent to which economic agents deviate from the optimal production state. Technically efficient firms can fully utilize production resources to maximize output or minimize costs. Technically inefficient firms, on the other hand, either produce output below potential levels given the resources or use excessive production resources for a given output level.







• Aigner et al. (1977) and Meeusen and van Den Broeck (1977) first propose SF models

$$egin{aligned} Y_{it} &= X_{it}^{\prime}eta + v_{it} - u_{it} \ v_{it} &\sim N(0,\sigma_v^2) \ u_{it} &\sim N^+(0,\sigma_u^2) \end{aligned}$$

v and u are independent.



Denote $\varepsilon = v - u$. The PDF of ε can be obtained as follows: $f(v, u) = f(v)f(u) = f(\varepsilon + u)f(u)$ $f(\varepsilon) = \int_{0}^{+\infty} f(\varepsilon + u)f(u)du$ $f(\varepsilon) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2\sigma^{2}}\varepsilon^{2}\right] * 2(1 - \Phi(\lambda\varepsilon/\sigma))$

 $\sigma^2 = \sigma_u^2 + \sigma_{v'}^2$ $\lambda = \sigma_u/\sigma_v$











Similar with linear regression models, we can get the log-likelihood function

$$\ln L = \sum_{it} \left[\ln(\frac{1}{\sigma}\phi(\varepsilon/\sigma)) + \ln\left[1 - \Phi\left(\varepsilon_{it}\lambda\sigma^{-1}\right)\right] + \ln(2) \right]$$

- Inefficiency estimates: E(u|arepsilon)
- efficiency estimates: $E(exp(-u)|\varepsilon)$



Recent development of Stochastic Frontier Models

- Fixed effects panel SF models
- SF models with endogenous variables
- SF models with crossectional dependence and spatial spillover



Inefficiency in technology refers to deviations from the common frontier, which may be partially due to individual heterogeneity. Fixed effects are commonly used to capture individual heterogeneity.







• Greene (2005): "true" fixed effect SF model

$$egin{aligned} Y_{it} &= X_{it}^{\prime}eta + lpha_i + v_{it} - su_{it} \ v_{it} &\sim N(0,\sigma_v^2) \ u_{it} &\sim N^+(0,\sigma_u^2) \end{aligned}$$

s=1 for production function; s=-1 for cost function



- Greene (2005) proposed the 'brute force' method to estimate the model.
- Remark: The incidental parameters problem does not affect the slope coefficients of a stochastic frontier model, while there is also evidence suggesting that the variance parameters are more likely to be affected when T is not large.



• The fixed effects can be removed by the first difference or within-tansfromation.

$$egin{aligned} ilde{Y}_{it} &= ilde{X}_{it}^{\prime}eta + ilde{v}_{it} - ilde{u}_{it} \ ilde{v}_{i.} &\sim MN(0,\Omega) \ ilde{u}_{i.} &\sim ? \end{aligned}$$

$$ilde{arepsilon}_{i.} = ilde{v}_{i.} - ilde{u}_{i.} \sim ?$$



• Wang and Ho (2010) assume $u_{it} = h(z_{it})u_i^*, u_i^* \sim N^+(0,\sigma_u^2)$

$$egin{aligned} ilde{u}_{i.} &= ilde{h}(z_{i.}) u_i^* \ f(ilde{v}_{i.}, ilde{u}_{i.} | z_{i.}) &= f(ilde{v}_{i.} | z_{i.}) f(ilde{u}_{i.} | z_{i.}) = f(ilde{arepsilon}_{i.} + ilde{u}_{i.} | z_{i.}) f(ilde{u}_{i.} | z_{i.}) \end{aligned}$$



• Chen et al. (2014) relax the WH model without assuming the scaling property of the inefficency term.

$$ilde{arepsilon}_{it} = Marepsilon_{it}
onumber \ M = I_T - rac{l_T l_T'}{T}$$

 $\tilde{\varepsilon}_{it}$ is a linear combinantion of ε_{it} .



• Chen et al. (2014) relax the WH model without assuming the scaling property of the inefficency term.

It is well known that the normal distribution has some important and convenient properties. (i) If two random variables are marginally normal and independent, they are jointly normal. (ii) If two random variables are jointly normal, they are marginally normal. (iii) If two random variables are jointly normal, the distribution of either one conditional on the other is normal. (iv) Linear combinations of jointly normal random variables are normal. The *CSN* family has analogous properties, as the following results show.



• Chen et al. (2014) relax the WH model without assuming the scaling property of the inefficency term.

Theorem 2.

$$\tilde{\varepsilon}_{i}^{*} \sim CSN_{T-1,T} \left(0_{T-1}, \sigma^{2} \left(I_{T-1} - \frac{1}{T} E_{T-1} \right), -\frac{\lambda}{\sigma} \begin{bmatrix} I_{T-1} \\ -1_{T-1}' \end{bmatrix}, 0_{T}, I_{T} + \frac{\lambda^{2}}{T} E_{T} \right),$$
(A)

$$\bar{\varepsilon}_i \sim CSN_{1,T} \left(0, \frac{\sigma^2}{T}, -\frac{\lambda}{\sigma} \mathbf{1}_T, \left(1 + \lambda^2 \right) I_T - \frac{\lambda^2}{T} E_T \right), \tag{B}$$

 $\tilde{\varepsilon}_i^*$ and $\bar{\varepsilon}_i$ are not independent (except when $\lambda = 0$). (C)



Endogeneity problems can arise in stochastic frontier models due to a couple of major reasons: First, the determinants of the frontier and the two-sided error term can be correlated. Secondly, the inefficiency term and two-sided error term can be correlated, or in particular, the determinants of the inefficiency can cause this correlation. Endogeneity in a stochastic frontier model would lead to inconsistent parameter estimates.



• Karakaplan, Mustafa U. (2017) and Karakaplan, Mustafa U. and Kutlu, Levent (2017)

The stochastic frontier panel data model is given as follows:



• Karakaplan, Mustafa U. (2017) and Karakaplan, Mustafa U. and Kutlu, Levent (2017)

By a Cholesky decomposition of the variance-covariance matrix of $(\tilde{\varepsilon}'_{it}, v_{it})'$, we can represent $(\tilde{\varepsilon}'_{it}, v_{it})'$ as follows:

$$egin{bmatrix} ilde{arepsilon}_{it} \ ilde{v}_{it} \end{bmatrix} = egin{bmatrix} I_p & 0 \ \sigma_v
ho' & \sigma_v \sqrt{1-
ho'
ho} \end{bmatrix} egin{bmatrix} ilde{arepsilon}_{it} \ ilde{w}_{it} \end{bmatrix}$$

where $\tilde{\varepsilon}_{it}$ and $\tilde{w}_{it} \sim \mathbf{N}(0,1)$ are independent.



 η

Handling endogenous variables

The frontier equation can be written as

$$y_{it} = x'_{yit}eta + \sigma_v
ho' ilde{arepsilon}_{it} + w_{it} - su_{it}$$

 $= x'_{yit}eta + (x_{it} - Z_{it}\delta)' \eta + e_{it}$
where $e_{it} = w_{it} - su_{it}, w_{it} = \sigma_v \sqrt{1 -
ho'
ho} ilde{w}_{it} = \sigma_w ilde{w}_{it}$ and $\eta = \sigma_w \Omega^{-1/2}
ho / \sqrt{1 -
ho'
ho}.$



- $(x_{it} Z_{it}\delta)'\eta$ is the biased corrected term. The random error v is decomposed into $(x_{it} - Z_{it}\delta)'\eta$ correlated with endogeneous variables x_{it} and w_{it} uncorrelated with endogeneous variables. This is the control function approach.
- Remark: The 2SLS method decomposes the endogenous variables x_{it} into \hat{x}_{it} and $(x_{it} \hat{x}_{it}) = [\varepsilon Z(Z'Z)^{-1}Z'\varepsilon]_{it}$. The two stage method can not be applied in SF models with endogeneous variables.

$$y_{it} = \hat{x}'_{yit}\beta + [\varepsilon - Z(Z'Z)^{-1}Z'\varepsilon]'_{it}\beta + v_{it} - su_{it}$$



• The frontier equation can be written as

$$y_{it} = x'_{yit}eta + \sigma_v
ho' ilde{arepsilon}_{it} + w_{it} - su_{it}$$

 $= x'_{yit}eta + (x_{it} - Z_{it}\delta)'\eta + e_{it}$
where $e_{it} = w_{it} - su_{it}, w_{it} = \sigma_v \sqrt{1 -
ho'
ho} ilde{w}_{it} = \sigma_w ilde{w}_{it}$ and
 $\eta = \sigma_w \Omega^{-1/2}
ho/\sqrt{1 -
ho'
ho}$.
 $f_{YX|Z} = f_{Y|XZ} f_{X|Z}$



Consider the case $u_{it} \sim N^+(0, \sigma_u^2)$ $\ln(f_{Y|XZ})_{it} = \ln(2) + \ln[\phi(e_{it}/\sigma)/\sigma] + \ln\left[1 - \Phi\left(e_{it}\lambda\sigma^{-1}\right)\right]$ $\sigma^2 = \sigma_u^2 + \sigma_w^2, \lambda = \sigma_u/\sigma_w$

 $f_{X|Z}$ is a multivariable normal distribution, the same as in the systems of linear regression models.

$$f(\mathbf{x}|\mathbf{z}) = rac{1}{\sqrt{(2\pi)^k |\mathbf{\Omega}|}} \exp\left(-rac{1}{2}(\mathbf{x} - \mathbf{z}' \boldsymbol{\delta})^T \mathbf{\Omega}^{-1}(\mathbf{x} - \mathbf{z}' \boldsymbol{\delta})
ight)$$



Crossectional dependence and Spatial spillover

Firms tend to concentrate in clusters, taking advantage of positive agglomeration externalities due to cooperation, shared ideas and emulation, resulting in increased productivity levels. Thus, producers cannot be regarded as isolated entities and the hypothesis of cross-sectional independence underlying the basic SF model must no longer be considered valid.

• Glass et al. (2016): SAR-SFA

$$egin{aligned} Y_{it} &=
ho \sum_{j} w_{ij} Y_{jt} + X'_{it} eta + v_{it} - s u_{it} \ v_{it} &\sim N(0, \sigma_v^2) \ u_{it} &\sim N^+(0, \sigma_u^2) \ Y_t &= (I -
ho W)^{-1} (X_t eta + v_t - s u_t) \ Denote \ Z_t &= X_t eta + arepsilon_t \ f_{Z_t \mid X_t} &= \Pi_i rac{2}{(2\pi\sigma^2)^N} e^{-rac{(Z_t - X'_t eta)'(Z_{it} - X'_{it} eta)}{2\sigma^2}} (1 - \Phi(\lambda(Z_{it} - X'_{it} eta) \sigma^{-1})) \ Z_t &= [I -
ho W] Y_t \ f_{Y_t \mid X_t} &= |I -
ho W| f_{Z_t \mid X_t} \ \log(f_{Y_t \mid X_t}) &= \log(|I -
ho W|) + \log(f(Z_t \mid X_t)) \end{aligned}$$





• Kultu et al. (2020): SAR-SFA with endogeneous variables

$$egin{aligned} y_{it} &=
ho \sum_{j} w_{ij} y_{jt} + oldsymbol{x}_{yit}' oldsymbol{eta} + v_{it} - \mathrm{s} u_{it} \ x_{it} &= Z_{it} \delta + arepsilon_{it} \ iggle x_{it} \end{bmatrix} \equiv iggl[oldsymbol{\Omega}^{-1/2} oldsymbol{arepsilon}_{it} \ v_{it} \end{bmatrix} &\sim \mathbf{N} \left(iggl[egin{smallmatrix} 0 \ 0 \end{bmatrix}, iggl[oldsymbol{I}_p & \sigma_v oldsymbol{
ho} \\ \sigma_v oldsymbol{
ho}' & \sigma_v^2 \end{bmatrix}
ight) \ u_{it} &= oldsymbol{h} \left(oldsymbol{x}_{uit}' arphi_u
ight) oldsymbol{u}_{it}^* \ s &= iggl\{ egin{smallmatrix} 1 & ext{for production functions} \\ -1 & ext{for cost functions} \end{array}
ight) \end{aligned}$$

北京友万信息科技有限公司 www.uone-tech.cn Spatial Stochastic Frontier Models (SSFM)

• Orea and Álvarez (2019) include the cross-sectional dependence of v_{it} and u_{it}

$$egin{aligned} &v_{it} = \gamma \sum_j w_{ij} v_{it} + v_{it}^st \ &u_{it} = au \sum_j w_{ij} u_{jt} + u_{it}^st \ &v_t = [I - \gamma W]^{-1} v_t^st \ &u_t = [I - au W]^{-1} u_t^st \ &arepsilon_t = v_t - u_t \ &f(v_t, u_t) = f(arepsilon_t + u_t, u_t) \end{aligned}$$

Remark: It is difficult to intergrate N-dimension random variables $u_t = (u_{1t}, \ldots, u_{Nt})'$ of $f(\varepsilon_t + u_t, u_t)$.



Orea and Álvarez (2019) assume $u_{it}^* = h(Z_{it})\tilde{u}_t$ where \tilde{u}_t is a scalar random variable which does not vary across individuals given time t. The N-dimension degenerates to one dimension.

The spatial SF model is a a transposed version of the WH model.

$$egin{aligned} & ilde{h}(Z_t) = [I - au W]^{-1} h(Z_t) \ & ilde{v}_t \sim MN(0, \sigma_v^2 [I - \gamma W]^{-1} ([I - \gamma W]^{-1})^T) \end{aligned}$$



- Galli (2022)
 - $\circ\,$ Global spatial spillovers Wy_t
 - $\circ\,$ Local spatial spillovers Wx_t
 - Cross-sectional dependence of error term
 - Cross-sectional dependence of inefficiency term

$$egin{aligned} Y_{it} &=
ho \sum_j w_{ij} Y_{jt} + X'_{it} eta + \sum_j W_j X'_{it} heta + v_{it} + s u_{it} \ Y_t &= (I -
ho W)^{-1} (X_t eta + \sum_j W_j X'_{it} heta + v_{it} + s u_{it}) \end{aligned}$$



• Galli (2022)

	yuv	xuv	yv	yu	y	xuv	xv	xu	uv	u	v
0		0				0	0	0	0	0	0
θ	0	0	0	0	0	Ŭ	Ū	Ŭ	0	0	0
γ				0	0			0		0	
au			0		0		0				0

Table 1: Specific models with restricted parameters



Estimating SF models in Stata

- Stata official: *frontier*, *xtfrontier*
- Belotti et al. (2013): sfcross, sfpanel
- Karakaplan (2017): *sfkk*
- Karakaplan (2018): *xtsfkk*
- Lian et al. (2013): *sftt*
- Kumbhakar et al. (2015) provides a practitioner's guide to stochastic frontier analysis with a suite of Stata commands (including sfmodel, sfpan, sf_fixeff, and sfprim).



spxtsfa: a new command for fitting spatial stochastic frontier models in the style of Orea and Álvarez (2019) and Galli(2022). We use Stata ml model routine with the method-*d0* evaluator to program the spxtsfa command.



• install from github

```
net install spxtsfa,
from("https://raw.githubusercontent.com/kerrydu/spxtsfa/main/spxtsfa-
statafiles/") replace
net get spxtsfa,
from("https://raw.githubusercontent.com/kerrydu/spxtsfa/main/spxtsfa-
statafiles/") replace
```

• install from gitee

net install spxtsfa, from("https://gitee.com/kerrydu/spxtsfa/raw/main/spxtsfastatafiles") replace net get spxtsfa, from("https://gitee.com/kerrydu/spxtsfa/raw/main/spxtsfastatafiles") replace



Estimation syntax

spxtsfa depvar [indepvars], uhet(varlist) [noconstant cost wy(wyspec)
wx(wxspec) wu(wuspec) wv(wvspec) normalize(norm_method)
wxvars(varlist) initial(matname) mlmodel(model_options)
mlsearch(search_options) mlplot mlmax(maximize_options) nolog
mldisplay(display_options) level(#) lndetmc(numlist)
te(newvarname) genwxvars delmissing constraints(constraints)]
Version syntax
spxtsfa , version

Replay syntax

spxtsfa [, level(#)]



wy(wyspec) specifies the spatial weight matrix for lagged dependent variable. The expression is wy(W_1 [$W_2 \dots W_T$] [,mata array]). By default, the weight matrices are Sp objects. mata declares weight matrices are mata matrices. If one weight matrix is specified, it assumes a time-constant weight matrix. For time-varying cases, T weight matrices should be specified in time order. Alternatively, using array to declare weight matrices are stored in an array. If only one matrix is stored in the specified array, the time-constant weight matrix is assumed. Otherwise, the keys of the array specify time information, and the values store time-specific weight matrices.



We first consider the yxuv-SAR model specified by the following data-generating process (DGP 1) with $i=1,\ldots,300$ and $t=1,\ldots,20$,

$$Y_{it} = 0.3 W_i Y_{.t} + 2 X_{it} + 0.3 W_i X_{.t} + ilde{v}_{it} - ilde{u}_{it}$$

where \tilde{v}_{it} and \tilde{u}_{it} are defined with $\gamma = 0.3$, $\tau = 0.3$, $\delta = 2$, $\sigma_u^2 = 0.2$ and $\sigma_v^2 = 0.2$. All the spatial matrices for the four spatial components are the same and time-invariant, created from a binary contiguity spatial weight matrix. We generate the exogenous variables X_{it} and Z_{it} from the standard normal distribution, respectively. With the sample generated by DGP 1, we can fit the model in the following syntax.

. use spxtsfa	_DGP1.dta							
. xtset id t								
Panel variable Time variable Delta	e: id (strongl e: t, 1 to 20 a: 1 unit	y balanced)					
. * importing . mata mata ma (loading w1[30	spatial weigh atuse spxtsfa_ 00,300])	t matrix f wmat1.mmat	rom spxts ,replace	fa_wmat1.	mmat			
. * fitting tl . spxtsfa y x	he model , uhet(z) noco	onstant wy	(w1,mata)	wx(w1,ma	uta) wu(w1,mat	a) wv(w1,mata)) wxvars(x)	nolog
Spatial front:	ier model(yxuv	-SAR)		Nu	mber of obs =	6,000		
Log likelihood	d = -1727.016			Wa Pr	ald chi2(2) = cob > chi2 =	118937.24 0.0000		
У	Coefficient	Std. err.	Z	P> z	[95% conf.	interval]		
frontier								
x	1.993915	.0065251	305.58	0.000	1.981126	2.006704		
W_x	.4435823	.0373189	11.89	0.000	.3704386	.516726		
uhet								
Z	2.000371	.0013412	1491.49	0.000	1.997742	2.002999		
/lnsigma2_u	-2.098104	.3163094	-6.63	0.000	-2.718059	-1.478149		
/lnsigma2_v	-1.637609	.018401	-89.00	0.000	-1.673674	-1.601544		
Wy _cons	.6605993	.0317043	20.84	0.000	. 5984599	.7227386		
	.5806681	.0318346	18.24	0.000	.5182735	.6430627		
Wv								
_cons	.5745429	.051903	11.07	0.000	.4728148	.676271		
sigma2_u	.1226888	.0388076	3.16	0.002	.0660027	.2280593		
sigma2_v	.1944444	.003578	54.34	0.000	.1875567	.2015851		
rho	.3187581	.0142397	22.39	0.000	.2905787	.3463849		
tau	.282414	.014646	19.28	0.000	.2534626	.3108598		
gamma	.2795936	.02392	11.69	0.000	.2320763	.3257792		





$$\eta = \left(\frac{1}{r_{\min}}\right)(1-p) + \left(\frac{1}{r_{\max}}\right)p$$
$$0 \le p = \frac{\exp\left(\delta_0\right)}{1+\exp\left(\delta_0\right)} \le 1$$



We consider the restricted model uv-SAR with time-varying spatial weight matrices as the second example. The DGP 2 is described as $Y_{it} = 1 + 2X_{it} + v_{it} - u_{it}, i = 1, ..., 300; t = 1, ..., 10.$

. use spxtsfa	_DGP2.dta					
. xtset id t						
Panel variabl Time variabl Delt	e: id (strongl e: t, 1 to 10 a: 1 unit	y balanced))			
<pre>. * importing . mata mata m (loading w1[3 w5[300,300],</pre>	<pre>spatial weigh atuse spxtsfa_ 00,300], w10[3 w6[300,300],</pre>	t matrices wmat2.mmat 00,300], w2 w7[300,300]	from spx1 ,replace 2[300,300]], w8[300	tsfa_wma , w3[30),300],	at2.mmat 00,300], w4[300 w9[300,300])	0,300],
<pre>. * fitting t . local w w1 spytsfa w w</pre>	he model w2 w3 w4 w5 w6	5 w7 w8 w9 u	w10		ficioncy) nol	or
Spatial front	, unet(2) wu(w , mata) w	v(w,mata		Number of obc	- 3.000
Log likelihoo	d = -1336.482	AR)			Wald chi2(1) Prob > chi2	= 3,000 = 43686.91 = 0.0000
У	Coefficient	Std. err.	z	P> z	[95% conf	. interval]
frontier						
x _cons	2.015288 .9415143	.0096419 .0160786	209.01 58.56	0.000	1.99639 .9100008	2.034186 .9730278
uhet z	2.000242	.0020671	967.66	0.000	1.99619	2.004293
/lnsigma2_u /lnsigma2_v	-2.006684 -1.300024	.4473506	-4.49 -49.98	0.000	-2.883475 -1.351002	-1.129893 -1.249045
Wu _cons	. 582383	.0031549	184.59	0.000	.5761995	.5885666
Wv _cons	.5374655	.0601775	8.93	0.000	.4195198	.6554113
sigma2_u	.1344337	.060139	2.24	0.025	.05594	.3230678
tau gamma	.2832028	.0014508	195.21 9.37	0.000	.2803569	.2860438 .3164261





• Spatial weight matrix can be created by *spmatrix* and *spwmatrix*

copy "https://gitee.com/kerrydu/spxtsfa/raw/main/city.zip" . unzipfile city.zip

cd ./city

```
spshape2dta province
```

spmatrix create contiguity M, normalize(row) rook

spmatrix create idistance W, normalize(row) replace

```
spmatrix matafromsp W id = M
```



 The likelihood function of spatial stochastic frontier models is complicated, and generally diffcult to obtain the optimal global solutions. Thus, good initial values would be helpful for fitting spatial stochastic models. Practitioners might fit the non-spatial stochastic models using fronteir and sfpanel commands to obtain the initial values of the parameters involved in the frontier and the scaling function and then use the mlplot option to search initial values for spatially-correlated parameters



```
use uvexample.dta

xtset id t

mata mata matuse spxtsfa_wmat.mmat,replace

frontier y x, noconst cost uhet(z)

mat b = e(b)

mat b = b[1,1],b[1,3],b[1,4],b[1,2]

mat b = b[1,1],b[1,3],b[1,4],b[1,2],0.2,0.2

spxtsfa y x, cost noconstant wu(w2,mata) wv(w1,mata) uhet(z) te(te) init(b) mlplot
```



Some issues

- "spxtsfa, version" which could potentially cause Stata to crash as it attempts to establish a connection with Github. Undocuemneted syntax: spxtsfa, version gitee
- Speed of spxtsfa is slow
- failure of convergence
- mata does not have routines of sparse matrix; consume a lot of memory for large spmatrix
- spxtfsa does not support vce(cluseter)



Thank you! Welcome your comments and suggestion